Quality and Reliability Chain Modeling for System Reliability Analysis of Multi-station Manufacturing Processes

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Abstract: This paper presents a generic methodology to integrate component reliability and product quality for multi-station manufacturing process reliability analysis. A new QR-chain model is developed to capture the complex QR interactions propagating throughout all stations. An analytical solution is obtained and its upper bound is provided to expedite practical implementation.

Notation

n	number of process variables
m	number of product quality characteristics
p	number of manufacturing system components
l	number of noise variables in the system
t	number of operation cycles, which is an operating time index
$\mathbf{X}(t)$	process variable vector at time t
$\mathbf{Y}(t)$	product quality characteristic vector at time t
$[\mathbf{b}]_i$	the i^{th} element of vector **b**
$F_{\mathbf{X}}$	Cdf of a random variable **X**
$\mathbf{A} \otimes \mathbf{B}$	Kronecker product of **A** and **B**
$\mathbf{x} \cdot {*}\mathbf{y}$	Hadamard product of two vectors **x** and **y** ($\mathbf{z} = \mathbf{x} \cdot {*}\mathbf{y} \Rightarrow [\mathbf{z}]_i = [\mathbf{x}]_i \cdot [\mathbf{y}]_i$)

1. Introduction

1.1 Problem Statement

Multi-station manufacturing processes (MMPs) are used in various industry sectors. Examples of MMPs include multi-assembly stations in automotive body assembly, transfer or progressive dies in stamping processes, and multiple pattern lithography operations in semiconductor manufacturing.

System reliability, as a critical performance index of an MMP, is defined as the probability that a manufacturing process performs its intended function under operating conditions for a specified period of time [1]. The intended function of an MMP should consider not only the manufacturing process uptime at each station, but also the produced product quality, as shown in Figure 1. In traditional reliability research literature [1, 2], system failures are determined only by system component failures due to their malfunctions or their degradations beyond the maximum acceptable amount of wear. The dependency of the manufacturing system reliability on the product quality was neglected in the system reliability modeling of a manufacturing process.

For a single station manufacturing process, the concept of the interaction between product quality and manufacturing system component reliability, called as QR-Co-Effect, has been studied in [3] for manufacturing process reliability modeling. As shown in Figure 1, the manufacturing system component degradation affects the outgoing product quality. Meanwhile the incoming raw material/part quality can affect the manufacturing system component reliability such as degradation rate and catastrophic failure rate.

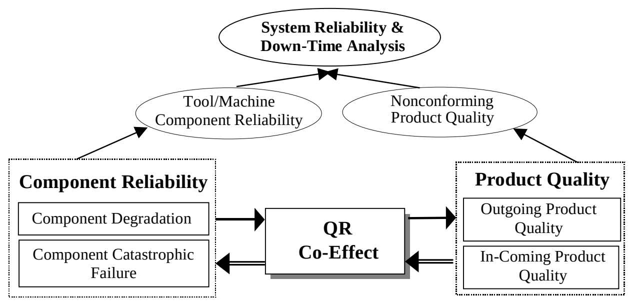


Figure 1. QR-Co-Effect between product quality and manufacturing system component reliability

In the single station situation [3], it is reasonable to assume that incoming product quality (raw material/parts) is time-independent, and so is the component catastrophic failure rate. Therefore, the probability of nonconforming product quality can be reasonably considered as s-independent of the probability of component catastrophic failure. However, this assumption will not hold anymore in an MMP due to the dependent propagation of product quality across stations. Therefore, development of an effective model to describe the quality and reliability dependency and its propagation throughout all stations is one of the major concerns in system reliability evaluation for an MMP.

1.2 QR-Chain Effect in an MMP

First the specific scope of the following terminologies used in this paper is clarified as follows:

- Component/Manufacturing system component — In this paper, component means the component of a manufacturing system, rather than the component of a product;
- Product — Both the input and output workpieces of each station of an MMP, including the intermediate parts and the final product;
- System catastrophic failure — System failure due to component catastrophic failures;
- System failure due to nonconforming products —

an event that the products are out of specifications. For MMPs, the specifications could be assigned to the intermediate or the final products.

In an MMP, each station consists of multiple components. To simplify the problem, all components in an MMP are assumed to be connected in series, that is, the catastrophic failure of any component will lead to the system catastrophic failure. In fact, all the developed models and methodologies in this paper can be easily adapted to general hybrid series/parallel systems. System reliability of an MMP is defined as the probability that neither the system catastrophic failure nor the failure due to nonconforming products occurs during a specific period of time.

The product variation is propagated in an MMP [4, 5]. Considering the QR-Co-Effect at each station, the variation propagation in product quality will lead to the propagation of the interaction between the manufacturing system component reliability and the product quality (as shown in Figure 2), which is called as the QR-Chain effect.

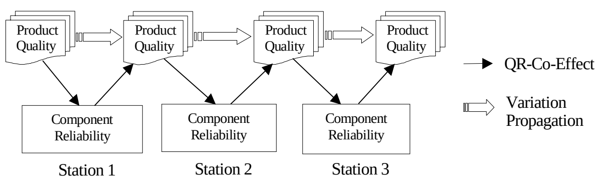


Figure 2. General concepts of the QR-Chain in MMPs

The QR-Chain effect can be observed in many MMPs, such as multistage machining process, multi-station assembly process, and transfer or progressive die stamping process. A cylinder head machining process shown in Figure 3 is used as an example here to illustrate the characteristics of the QR-Chain effect [6]: there are two stations consisting of drilling a hole in a cylinder head (station I) and then tapping a thread on it afterwards (station II). In station I, the material properties of the incoming workpiece (taken as the product quality) will have a significant impact on the wear and breakage rate of the drill (taken as the component reliability). The drill condition further impacts on the quality of the hole drilled in this station in terms of size, straightness, and orientation, etc. In the next tapping station (station II), those drilled hole quality characteristics of station I are essential factors affecting the thread quality of station II and the wear and breakage rate of the taper tool. Therefore, the QR-Co-Effect at station I has propagated to station II.

From this example, the characteristics of the QR-Chain effect in an MMP can be summarized as: the stochastic degradation of the manufacturing system components causes the stochastic deterioration of the outgoing product quality; furthermore, the dependent propagation of product quality across stations causes the dependency of the manufacturing system component failures and the system failure due to nonconforming products.

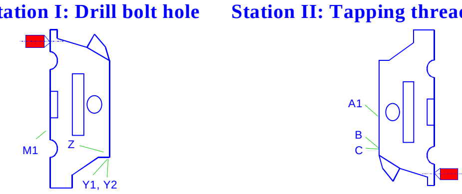


Figure 3. QR-Chain in a machining process

This paper is organized as follows: a new system reliability model, a QR-Chain model, is proposed in Section 2 to study the QR-Chain effect. In Section 3, an analytical solution for system reliability is obtained based on the proposed QR-Chain model. An upper bound of the system reliability is derived. In Section 4, an example is used to illustrate the analysis procedures and the effectiveness of the proposed methodology.

2. QR-Chain Modeling

Based on the characteristics of the QR-Chain effect, key elements of the QR-Chain model will be discussed in this section.

2.1 Relationship between Component Performance and Product Quality

The product quality in an MMP is generally affected by the state of multiple manufacturing system components. In this paper, the component state is described by *process variables*. Another type of variables affecting the product quality is the *noise variables*. Examples of noise variables include random variations of raw material quality and random environmental variations. In general, noise variables randomly change from one operation to the next. Considering further the interaction between the process variables and the noise variables, the following general linear model, which is called as the *process model* in this paper, is assumed for the product quality characteristics $Y_j(t)$

$$Y_j(t) = \eta_j + \boldsymbol{\alpha}_j^T \mathbf{X}(t) + \boldsymbol{\beta}_j^T \mathbf{z}_t + \mathbf{X}(t)^T \boldsymbol{\Gamma}_j \mathbf{z}_t,\ \ j = 1,2,\ldots,m \quad (1)$$

where $\mathbf{X}(t) \in R^n$ is the vector of process variables in the MMP, $\mathbf{z}_t = [z_{1t}, z_{2t}, \ldots, z_{lt}]^T \in R^l$ is the vector of noise variables, with mean E($\mathbf{z}_t$) and covariance matrix Cov($\mathbf{z}_t$) independent of the time index, $\eta_j,\ j = 1,2,\ldots,m$ are constants, $\boldsymbol{\alpha}_j$ and $\boldsymbol{\beta}_j$ are vectors characterizing the effects of $\mathbf{X}(t)$ and $\mathbf{z}_t$, and $\boldsymbol{\Gamma}_j$ is a matrix characterizing the interaction effects between $\mathbf{X}(t)$ and $\mathbf{z}_t$.

Remarks: The process model (1) could be obtained from physical process models or by using the design of experiments (DOE) since the process model (1) has the same structure as the response model in robust parameter design [7].

2.2 System Component Degradation

The process variables $\mathbf{X}(t)$ in (1) change over time due to system component degradation. Suppose that the time axis is divided into contiguous and uniform intervals of length h, and that the successive endpoints of the intervals are denoted by h, $2h$, $3h$, …, kh, …. The process variable vector $\mathbf{X}(t_k)\in R^n$ is used to represent the degradation state of system components at time $t_k\equiv kh$, $k=1,2,\ldots$. The production mission time is $t_K = Kh$. The following Gauss-Markov model is assumed:

$$\mathbf{X}(t_{k+1}) = \mathbf{A}_k \mathbf{X}(t_k) + \mathbf{G}_k \boldsymbol{\varepsilon}_k,\ k = 0, 1, 2, \ldots \qquad (2)$$

where $\boldsymbol{\varepsilon}_k \in R^g$; $\mathbf{A}_k$ and $\mathbf{G}_k$ are possibly time-varying, known matrices of appropriate dimension; $\{\boldsymbol{\varepsilon}_k, k \ge 1\}$ are i.i.d. and Gaussian, with $\boldsymbol{\varepsilon}_k \sim N(\boldsymbol{\mu}_\varepsilon, \mathbf{Q})$; $\mathbf{X}(t_0) \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ represents the initial value of the process variables. Eq. (2) can be considered as a multivariate discrete approximation to a diffusion process. Without loss of generality, in this paper the components are in the ideal state if $\mathbf{X}(t)=\mathbf{0}$. It is also reasonable to assume increasing wear in our model, that is, for any $1\le j\le n$, $\Pr\{[\mathbf{X}(t_0)]_j<0\}$ and $\Pr\{[\mathbf{X}(t_{k+1}) - \mathbf{X}(t_k)]_j < 0\}$ can be ignored. In fact, decreasing (or negative) wear is not meaningful in many applications [2].

2.3 Product Quality Assessment and System Failure due to Nonconforming Products

Based on the degradation model assumed in the previous section, (1) can be rewritten as

$$\begin{aligned} &Y_j(t) = \eta_j + \boldsymbol{\alpha}_j^T \mathbf{X}(t_k) + \boldsymbol{\beta}_j^T \mathbf{z}_t + \mathbf{X}(t_k)^T \boldsymbol{\Gamma}_j \mathbf{z}_t, \\ &t_k \le t < t_{k+1},\ j = 1,2,\ldots,m \end{aligned} \qquad (3)$$

In this paper, the product quality is assessed by the mean squared deviation of the quality characteristics from the target. Following this concept, under given component degradation state $\mathbf{X}(t_k)$, the j^{th} quality index $q_j(t)$ can be defined as

$$\begin{aligned} &q_j(t \mid \mathbf{X}(t_k)) \equiv E((Y_j(t) - \gamma_j)^2 \mid \mathbf{X}(t_k)) = \\ &Var(Y_j(t) \mid \mathbf{X}(t_k)) + E^2((Y_j(t) - \gamma_j) \mid \mathbf{X}(t_k)), t_k \le t < t_{k+1} \end{aligned} \qquad (4)$$

where γ_j is the target value for the j^{th} quality characteristic. Based on the meaning of the ideal state of the components, the mean of the quality characteristic achieves the target value when $\mathbf{X}(t_k)=\mathbf{0}$. Thus from (3),

$$\gamma_j = E(Y_j(t) \mid \mathbf{X}(t_k) = \mathbf{0}) = \eta_j + \boldsymbol{\beta}_j^T E(\mathbf{z}_t) \qquad (5)$$

and

$$\begin{aligned} &Var(Y_j(t) \mid \mathbf{X}(t_k)) = \boldsymbol{\beta}_j^T \operatorname{cov}(\mathbf{z}_t)\boldsymbol{\beta}_j \\ &+ (\mathbf{X}(t_k)^T \boldsymbol{\Gamma}_j) \operatorname{cov}(\mathbf{z}_t)(\boldsymbol{\Gamma}_j^T \mathbf{X}(t_k)) \end{aligned}, \qquad (6)$$

From (3) and (5),

$$\begin{aligned} &E^2((Y_j(t) - \gamma_j) \mid \mathbf{X}(t_k)) = (\boldsymbol{\alpha}_j^T \mathbf{X}(t_k) + \mathbf{X}(t_k)^T \boldsymbol{\Gamma}_j E(\mathbf{z}_t))^2 \\ &= \mathbf{X}(t_k)^T (\boldsymbol{\alpha}_j + \boldsymbol{\Gamma}_j E(\mathbf{z}_t))(\boldsymbol{\alpha}_j + \boldsymbol{\Gamma}_j E(\mathbf{z}_t))^T \mathbf{X}(t_k) \end{aligned} \qquad (7)$$

From (4), (6), and (7), $q_j(t)$ can be written as a quadratic function of $\mathbf{X}(t_k)$, thus we have

$$\begin{aligned} &q_j(t \mid \mathbf{X}(t_k)) = \mathbf{X}(t_k)^T \mathbf{B}_j \mathbf{X}(t_k) + d_j, \\ &j = 1,2,\ldots,m, t_k \le t < t_{k+1} \end{aligned} \qquad (8)$$

where $\mathbf{B}_j = \boldsymbol{\Gamma}_j \operatorname{cov}(\mathbf{z}_t)\boldsymbol{\Gamma}_j^T + (\boldsymbol{\alpha}_j + \boldsymbol{\Gamma}_j E(\mathbf{z}_t))(\boldsymbol{\alpha}_j + \boldsymbol{\Gamma}_j E(\mathbf{z}_t))^T$ and $d_j = \boldsymbol{\beta}_j^T \operatorname{cov}(\mathbf{z}_t)\boldsymbol{\beta}_j$. It is easy to see that $\mathbf{B}_j$ in (8) is positive semidefinite and $d_j \ge 0$.

For each quality index, there is a threshold value based on the product design specifications. Suppose the threshold for the j^{th} quality index is a_j, we define E_t^q as the event that all quality indexes are within the specification by time t, i.e.

$$E_t^q \equiv \bigcap_{j=1}^{m} \left(q_j(\tau) \le a_j, \forall 0 \le \tau \le t\right)$$

2.4 Component Catastrophic Failure and Its Induced System Catastrophic Failure

We assume that Pr{component i fails at operation t+1| it works at operation t, $\mathbf{Y}(t)$} is equal to

$$\lambda_{0i} + \mathbf{s}_i^T ((\mathbf{Y}(t) - \boldsymbol{\gamma}) \cdot * (\mathbf{Y}(t) - \boldsymbol{\gamma})), i = 1,2,\ldots, p$$

where $\mathbf{s}_i \in R^m$, called as the QR-coefficient in this paper, has only nonnegative elements; λ_{0i} is the initial fixed failure rate, $\boldsymbol{\gamma} \equiv [\gamma_1 \quad \gamma_2 \quad \ldots \quad \gamma_m]^T$, and $\mathbf{Y}(t) \equiv [Y_1(t) \quad Y_2(t) \quad \ldots \quad Y_m(t)]^T$. It is worth to note that with the failure rate achieving the minimum at $\mathbf{Y}(t) = \boldsymbol{\gamma}$ (quality characteristics right on the target) and the assumption that the effects of different product characteristics on the component catastrophic failure are independent, the quadratic relationship above can be considered as a second order approximation of a general functional relationship based on the Taylor series. Let $\lambda_i(t)$ denote the catastrophic failure rate of component i at operation t. By taking expectation on the noise variables in $\mathbf{Y}(t)$ and further using the definition of the failure rate (or hazard rate), $\lambda_i(t)$ can be written as

$$\begin{aligned} \lambda_i(t) &= \Pr\{\text{component } i \text{ fails at } t+1 \mid \text{it works at } t, \mathbf{X}(t_k)\} \\ &= \mathrm{E}(\lambda_{0i} + \mathbf{s}_i^T ((\mathbf{Y}(t) - \boldsymbol{\gamma}) \cdot * (\mathbf{Y}(t) - \boldsymbol{\gamma}) \mid \mathbf{X}(t_k)) \\ &= \lambda_{0i} + \mathbf{s}_i^T \mathbf{q}(t \mid \mathbf{X}(t_k)) \end{aligned} \qquad (9)$$

where $\mathbf{q}(t \mid \mathbf{X}(t_k)) = [q_1(t \mid \mathbf{X}(t_k)) \quad \ldots \quad q_m(t \mid \mathbf{X}(t_k))]^T$.

In this paper, E_t^c is used to denote the event that catastrophic failures never occurred at any of the p system components by time t.

3. System Reliability Evaluation

Based on the notations introduced in the last section, the system reliability at the production mission time t_K is $R(t_K) = \Pr\{E_{t_K}^q \cap E_{t_K}^c\}$.

3.1 Challenges in System Reliability Evaluation

The complex interactions among different elements of the QR-Chain model lead to three major challenges in system reliability evaluation: (1) Dependency between E_t^q and E_t^c; (2) Dependency among catastrophic failures of system components; (3) *Doubly stochastic property*: From (8) and (9), the catastrophic failure rate $\lambda_i(t)$ depends on another stochastic process $\{\mathbf{X}(t_k), k=1,2,...\}$. In the literature, this kind of process is called as doubly stochastic Poisson process [8].

3.2 System Reliability Evaluation of MMPs

The following three analysis steps are proposed to evaluate the QR-Chain model:

Step 1. Conditioning on the degradation path of each component, the component catastrophic failure process, which is a doubly stochastic Poisson process, will become a Non-homogeneous Poisson Process. And the dependency of E_t^q and E_t^c can also be removed.

Step 2. Uncondition on the degradation paths by implementing expectation to the conditional system reliability calculated in Step 1.

Step 3. Reorganize the integrand into the form of the p.d.f. of a Gaussian random variable.

3.2.1 Step 1

Conditioning on $\mathbf{X}_K \equiv [\mathbf{X}^T(t_0) \quad \mathbf{X}^T(t_1) \quad \dots \quad \mathbf{X}^T(t_K)]^T$, the system reliability

$$R(t_K \mid \mathbf{X}_K) = \Pr\{E_{t_K}^c \mid \mathbf{X}_K\}\Pr\{E_{t_K}^q \mid \mathbf{X}_K\}. \qquad (10)$$

where $\Pr\{E_{t_K}^c \mid \mathbf{X}_K\}$ and $\Pr\{E_{t_K}^q \mid \mathbf{X}_K\}$ can be calculated by the following results.

***Result* 1**. Let $\mathbf{d} \equiv [d_1 \quad d_2 \quad \dots \quad d_m]^T$ and $c \equiv \sum_{i=1}^{p}\left(\lambda_{0i} + \mathbf{s}_i^T\mathbf{d}\right)$. There exists a positive semidefinite matrix $\mathbf{U}_K$**,** such that

$$\Pr\{E_{t_K}^c \mid \mathbf{X}_K = \mathbf{x}_K\} = \exp\left(-\left(ct_K + \mathbf{x}_K^T\mathbf{U}_K\mathbf{x}_K\right)\right). \quad \blacktriangleleft$$

The proof of Result 1 can be found in Appendix A1.

***Result* 2**. $\mathbf{\Omega}$ is a domain in $R^{(K+1)\times n}$ s.t. $\mathbf{x}_K \in \mathbf{\Omega} \Leftrightarrow \bigcap_{k=0}^{K}\bigcap_{j=1}^{m}\left\{\mathbf{x}^T(t_k)\mathbf{B}_j\mathbf{x}(t_k) \le a_j - d_j\right\}$. Let

$$I_K \equiv \begin{cases} 1, & \text{if } \mathbf{x}_K \in \mathbf{\Omega} \\ 0, & \text{otherwise} \end{cases}, \text{ then } \Pr\{E_{t_K}^q \mid \mathbf{X} = \mathbf{x}\} = I_K. \quad \blacktriangleleft$$

This result is obvious from (8).

From result 1, result 2, and (10), the conditional system reliability is

$$R(t_K \mid \mathbf{X}_K = \mathbf{x}_K) = \exp\left(-\left(ct_K + \mathbf{x}_K^T\mathbf{U}_K\mathbf{x}_K\right)\right)I_K \quad (11)$$

3.2.2 Step 2

Unconditioning on $\mathbf{X}_K$ by implementing expectation to (11), the system reliability becomes

$$R(t_K) = \mathop{\mathrm{E}}_{\mathbf{X}_K}\left[R(t_K \mid \mathbf{X}_K = \mathbf{x}_K)\right] = \int R(t_K \mid \mathbf{X}_K = \mathbf{x}_K)dF_{\mathbf{X}_K}(\mathbf{x}_K)$$

Further from (11),

$$\begin{aligned} R(t_K) &= \int \exp\left(-\left(ct_K + \mathbf{x}_K^T\mathbf{U}_K\mathbf{x}_K\right)\right)I_K dF_{\mathbf{X}_K}(\mathbf{x}_K) \\ &= \int_{\mathbf{x}_K \in \mathbf{\Omega}} \exp\left(-\left(ct_K + \mathbf{x}_K^T\mathbf{U}_K\mathbf{x}_K\right)\right)dF_{\mathbf{X}_K}(\mathbf{x}_K) \end{aligned} \qquad (12)$$

Since $\mathbf{X}(t_0)$, $\mathbf{X}(t_1)$, …, $\mathbf{X}(t_K)$ are jointly Gaussian,

$$\mathbf{X}_K \sim N(\boldsymbol{\mu}_K, \mathbf{\Sigma}_K) \qquad (13)$$

where $\boldsymbol{\mu}_K$ and $\mathbf{\Sigma}_K$ can be easily obtained from (2). Substitute $F_{\mathbf{X}_K}(\mathbf{x}_K)$ in (12) based on (13), the system reliability becomes

$$R(t_k) = \exp(-ct_K)\int_{\mathbf{x}_K \in \mathbf{\Omega}} \frac{1}{(2\pi)^{\frac{n(K+1)}{2}}\left|\mathbf{\Sigma}_K\right|^{\frac{1}{2}}}\exp\{-(\mathbf{x}_K^T\mathbf{U}_K\mathbf{x}_K + \frac{1}{2}(\mathbf{x}_K - \boldsymbol{\mu}_K)^T\mathbf{\Sigma}_K^{-1}(\mathbf{x}_K - \boldsymbol{\mu}_K))\}d\mathbf{x}_K \qquad (14)$$

3.2.3 Step 3

We can transform the integrand in (14) to the form of the density function of another multivariate Gaussian r.v. $\tilde{\mathbf{X}}_K$ by following results.

***Lemma* 1**. There exists a positive definite matrix $\tilde{\mathbf{\Sigma}}_K$, a vector $\tilde{\boldsymbol{\mu}}_K$, and a scalar s_K>0 such that

$$\begin{aligned} &\mathbf{x}_K^T\mathbf{U}_K\mathbf{x}_K + \frac{1}{2}(\mathbf{x}_K - \boldsymbol{\mu}_K)^T\mathbf{\Sigma}_K^{-1}(\mathbf{x}_K - \boldsymbol{\mu}_K) \\ &= \frac{1}{2}(\mathbf{x}_K - \tilde{\boldsymbol{\mu}}_K)^T\tilde{\mathbf{\Sigma}}_K^{-1}(\mathbf{x}_K - \tilde{\boldsymbol{\mu}}_K) + s_K \end{aligned} \qquad (15)$$

The proof of Lemma 1 can be found in appendix A2. Based on the lemma above, the integrand can be transformed into the form of a multivariate Gaussian p.d.f. as follows.

***Result* 3**. Let $\tilde{\mathbf{X}}$ be a $n(K+1)$ dimension Gaussian r. v. whose p.d.f. is

$$f_{\tilde{\mathbf{X}}_K}(\mathbf{x}_K) = \frac{1}{(2\pi)^{\frac{n(K+1)}{2}}\left|\tilde{\mathbf{\Sigma}}_K\right|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left(\mathbf{x}_K - \tilde{\boldsymbol{\mu}}_K\right)^T\tilde{\mathbf{\Sigma}}_K^{-1}\left(\mathbf{x}_K - \tilde{\boldsymbol{\mu}}_K\right)\right\}$$

Then the system reliability can be written as

$$R(t_K) = \exp(-ct_K)\frac{\exp(-s_K)}{\left|\mathbf{\Sigma}_K\right|^{\frac{1}{2}}}\left|\tilde{\mathbf{\Sigma}}_K\right|^{\frac{1}{2}}\int_{\mathbf{x}_K \in \mathbf{\Omega}} dF_{\tilde{\mathbf{X}}_K}(\mathbf{x}_K) \quad (16)$$

The proof of (16) can be obtained by reorganizing the exponential part of the integrand in (14) based on Lemma 1.

The multi-fold integral in (16) can be calculated by the following simulation: Suppose N_s Gaussian random variables are generated with mean $\tilde{\boldsymbol{\mu}}_K$ and variance

$\tilde{\mathbf{\Sigma}}_K$, among which N_0 generated random variables fall in the quality constraint $\mathbf{\Omega}$. Then N_0/N_s is an estimate of the integral $\int_{\mathbf{x}_K\in\mathbf{\Omega}} dF_{\tilde{\mathbf{X}}_K}(\mathbf{x}_K)$.

3.2.4 Self-Improvement of Product Quality and the Upper Bound of System Reliability

Note that the dimension of the integral in (16) is $n(K+1)$, which is generally very large. Especially the integral dimension depends on the production mission time t_K. It generally takes enormous computation resources to generate random variables with such a large dimension. However, we can significantly save the computation resources for the situation that the product quality does not have self-improvement, which will be discussed as follows.

In (8), $q_j(t)$ is not monotone non-decreasing over time t in general. If $q_j(t)$ is decreasing at t, we say that the product quality is self-improved at that time. If the product of an MMP does not have self-improvement at any time, that is, $q_j(t)$ is monotone non-decreasing, $\mathbf{x}(t_K)\in\mathbf{\Omega}_K \Leftrightarrow I_K=1$, where we say $\mathbf{x}(t_K)\in\mathbf{\Omega}_K$ if $\bigcap_{j=1}^{m}\left\{\mathbf{x}^T(t_K)\mathbf{B}_j\mathbf{x}(t_K)\le a_j-d_j\right\}$.

Thus, (16) can be rewritten as

$$R(t_K)=\exp(-ct_K)\frac{\exp(-s_K)}{\left|\mathbf{\Sigma}_K\right|^{\frac{1}{2}}}\left|\tilde{\mathbf{\Sigma}}_K\right|^{\frac{1}{2}}\int_{\mathbf{x}(t_K)\in\mathbf{\Omega}_K} dF_{\tilde{\mathbf{X}}_K}(\mathbf{x}_K) \qquad (17)$$

The integral domain of (17) depends only on $\mathbf{x}(t_K)$, not on $\mathbf{x}(t_{K-1})$, …, $\mathbf{x}(t_0)$. So the integral can be calculated based on the marginal distribution of $\tilde{\mathbf{X}}(t_K)$, which is the last n elements of $\tilde{\mathbf{X}}_K$. As $\tilde{\mathbf{X}}_K$ is multivariate Gaussian, the marginal distribution for $\tilde{\mathbf{X}}(t_K)$ can be directly obtained from the distribution of $\tilde{\mathbf{X}}_K$. Following this way, the $n(K+1)$ fold integral can be finally reduced to an n fold integral as follows.

***Result* 4**. The system reliability can be written as

$$R(t_K)=\exp(-ct_K)\frac{\exp(-s_K)}{\left|\mathbf{\Sigma}_K\right|^{\frac{1}{2}}}\left|\tilde{\mathbf{\Sigma}}_K\right|^{\frac{1}{2}}\int_{\mathbf{x}(t_K)\in\mathbf{\Omega}_K} dF_{\tilde{\mathbf{X}}(t_K)}(\mathbf{x}(t_K)) \qquad (18)$$

The proof of result 4 can be found in Appendix A3. How to obtain the distribution of $\tilde{\mathbf{X}}(t_K)$ in (18) is discussed in Appendix A4.

In general cases when the product quality may have self-improvement, we have $I_K=1\Rightarrow\mathbf{X}(t_K)\in\mathbf{\Omega}_K$, but the converse ($\mathbf{X}(t_K)\in\mathbf{\Omega}\Rightarrow I_K=1$) is not true. So (18) tends to overestimate the system reliability in general cases. However, (18) can be generally treated as an upper bound estimation of the system reliability, which is much easier to evaluate than the exact system reliability in (16).

4. Numerical Analysis

A manufacturing process with three tooling components allocated in three stations, as shown in Figure 4, is considered in the numerical analysis. The performance of these three components is described by the process variables $\mathbf{X}(t)\equiv[X_1(t)\quad X_2(t)\quad X_3(t)]^T$. Six noise variables z_{it}, i=1,2,3,4,5,6, whose distributions are designated in Figure 4, interact with the process variables and impact the product quality. There are four product quality characteristics $Y_i(t), i=1,2,3,4$ as shown in Figure 4.

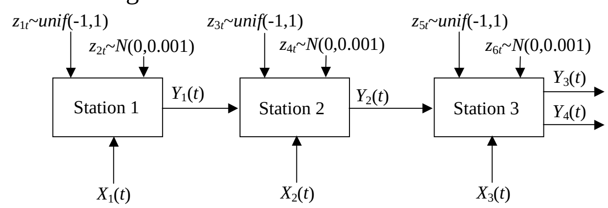


Figure 4. An example of a three station manufacturing process

The process model of this MMP is represented as:

$$Y_1(t)=0.274X_1(t)+z_{2t}-0.483X_1(t)z_{1t} \qquad (19)$$

$$\begin{aligned}Y_2(t)&=0.372Y_1(t)+0.365X_2(t)+0.564z_{4t}+X_2(t)z_{3t}\\&=0.102X_1(t)+0.365X_2(t)+0.372z_{2t}+\\&\quad 0.564z_{4t}-0.180X_1(t)z_{1t}+X_2(t)z_{3t}\end{aligned} \qquad (20)$$

$$\begin{aligned}Y_3(t)&=0.628Y_2(t)+0.303X_3(t)-0.722z_{6t}+0.775X_3(t)z_{5t}\\&=0.064X_1(t)+0.229X_2(t)+0.303X_3(t)\\&\quad+0.234z_{2t}+0.354z_{4t}-0.722z_{6t}-0.113X_1(t)z_{1t}\\&\quad+0.628X_2(t)z_{3t}+0.775X_3(t)z_{5t}\end{aligned} \qquad (21)$$

$$Y_4(t)=0.426X_3(t)-0.703X_3(t)z_{5t} \qquad (22)$$

The component degradation model is

$$\mathbf{X}(t_{k+1})=\mathbf{X}(t_k)+\boldsymbol{\varepsilon}_k,\ k=0,\ 1,\ 2,\ \ldots$$

where $\boldsymbol{\varepsilon}_k\sim N(\boldsymbol{\mu}_\varepsilon,\mathbf{Q})$, $\boldsymbol{\mu}_\varepsilon=10^{-3}\times[2\quad 2\quad 2]^T$, $\mathbf{Q}=2.5\times10^{-7}\times\mathbf{I}_{(3\times3)}$, and h=1000.

As in (8), product quality is assessed as

$$q_j(t\,|\,\mathbf{X}(t_k))=\mathbf{X}(t_k)^T\mathbf{B}_j\mathbf{X}(t_k)+d_j,\ j=1,2,3,4,\ t_k\le t<t_{k+1}$$

$\mathbf{B}_j$ and d_j can be obtained from (8) and the process model (19)-(22) as

$$\mathbf{B}_1=\begin{bmatrix}0.308&0&0\\0&0&0\\0&0&0\end{bmatrix},\ \mathbf{B}_2=\begin{bmatrix}0.043&0.037&0\\0.037&1.133&0\\0&0&0\end{bmatrix},$$

$$\mathbf{B}_3=\begin{bmatrix}0.017&0.015&0.019\\0.015&0.447&0.069\\0.019&0.069&0.692\end{bmatrix},\ \mathbf{B}_4=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0.676\end{bmatrix},$$

$d_1=10^{-3}$, $d_2=0.457*10^{-3}$, $d_3=0.701*10^{-3}$, and $d_4=0$.

Due to the QR-Chain effect, the component catastrophic failure rates are affected by product quality. Let $\lambda_0=6\times10^{-7}$ and $s=3\times10^{-4}$. It is assumed that

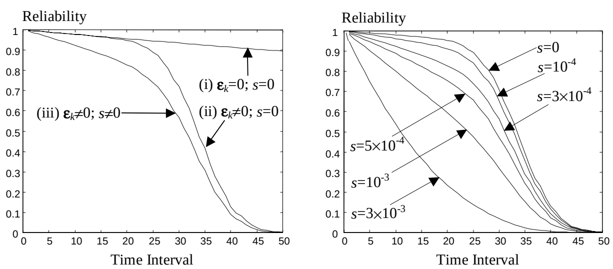
$$\lambda_1(t) = \lambda_0\,;\ \lambda_2(t) = \lambda_0 + s\cdot q_1(t)\,;\ \lambda_3(t) = \lambda_0 + s\cdot q_2(t)\,.$$

- System reliability evaluation

Since all elements of $\mathbf{B}_j$, $j = 1, 2, 3, 4$, are nonnegative, it is easy to see that the product quality of this manufacturing process does not have self-improvement. Thus, Result 4 can be used to evaluate the system reliability. The reliability plots calculated with Result 4 are shown in Figure 5. Three cases as follows are compared in Figure 5(a):

(i): $\boldsymbol{\varepsilon}_k = 0, s = 0$; (ii): $\boldsymbol{\varepsilon}_k \neq 0, s = 0$; (iii): $\boldsymbol{\varepsilon}_k \neq 0, s \neq 0$,

Compared with case (iii), both case (i) and case (ii) tend to overestimate the system reliability because they incorrectly neglect the QR-Chain effect existing in an MMP. A sensitivity analysis for different values of the QR-coefficient s is given in Figure 5(b).



(a) System reliability (b) Sensitivity analysis for different s

Figure 5. System reliability and sensitivity analysis

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Appendix

A1. Proof of Result 1

Conditioning on $\mathbf{X}_K$, we have,

$$\Pr\left\{E_{t_K}^c \mid \mathbf{X}_K\right\} = \exp\left(-\sum_{i=1}^{p}\left(\sum_{k=0}^{K-1}\lambda_i(t_k)h\right)\right) \qquad \text{(A1)}$$

Let $\mathbf{U}' \equiv h\sum_{i=1}^{p}\sum_{j=1}^{m}[\mathbf{s}_i]_j\mathbf{B}_j$ and $\mathbf{U}_K \equiv \begin{bmatrix}\mathbf{I}_{(K\times K)}\otimes\mathbf{U}' & \mathbf{0}_{(nK\times n)}\\ \mathbf{0}_{(n\times nK)} & \mathbf{0}_{(n\times n)}\end{bmatrix}$

Since the elements of $\mathbf{s}_i$ are nonnegative and $\mathbf{B}_j$, j=1,2,…, m are positive semidefinite, $\mathbf{U}'$ is positive semidefinite and $\mathbf{U}_K$ is positive semidefinite. From (8) and (9) and basic algebra manipulations,

$$\Pr\left\{E_{t_K}^c \mid \mathbf{X}_K\right\} = \exp\left(-\left(ct_K + \mathbf{X}_K^T\mathbf{U}_K\mathbf{X}_K\right)\right)\,.$$

A2. Proof of Lemma 1

$\boldsymbol{\Sigma}_K^{-1}$ is positive definite and $\mathbf{U}_K$ is positive semidefinite (from Result 1) $\Rightarrow \tilde{\boldsymbol{\Sigma}}_K^{-1} \equiv 2\mathbf{U}_K + \boldsymbol{\Sigma}_K^{-1}$ is positive definite. Let $\tilde{\boldsymbol{\mu}}_K \equiv \left(\mathbf{U}_K + \boldsymbol{\Sigma}_K^{-1}/2\right)^{-1}\left(\boldsymbol{\Sigma}_K^{-1}/2\right)\boldsymbol{\mu}_K$, $\tilde{\boldsymbol{\Sigma}}_K^{-1} \equiv 2\mathbf{U}_K + \boldsymbol{\Sigma}_K^{-1}$ and

$$s_K \equiv \boldsymbol{\mu}_K^T\left[\mathbf{U}_K^T\left(\mathbf{U}_K + \left(\boldsymbol{\Sigma}_K^{-1}/2\right)\right)^{-T}\left(\boldsymbol{\Sigma}_K^{-1}/2\right)\right]\boldsymbol{\mu}_K > 0$$

By substituting $\tilde{\boldsymbol{\mu}}_K, \tilde{\boldsymbol{\Sigma}}_K^{-1}$ and s_K, the lemma is followed.

A3. Proof of Result 4

From (17),

$$R(t_K) = \exp(-ct_K)\frac{\exp(-s_K)}{\left|\boldsymbol{\Sigma}_K\right|^{\frac{1}{2}}}\left|\tilde{\boldsymbol{\Sigma}}_K\right|^{\frac{1}{2}}\int_{\mathbf{x}(t_K)\in\boldsymbol{\Omega}_K} dF_{\tilde{\mathbf{X}}_K}(\mathbf{x}_K)$$

$$= \exp(-ct_K)\frac{\exp(-s_K)}{\left|\boldsymbol{\Sigma}_K\right|^{\frac{1}{2}}}\left|\tilde{\boldsymbol{\Sigma}}_K\right|^{\frac{1}{2}}\int \Pr\{\tilde{\mathbf{X}}(t_K)\in\boldsymbol{\Omega}_K \mid \tilde{\mathbf{X}}_K = \mathbf{x}_K\}\,dF_{\tilde{\mathbf{X}}_K}(\mathbf{x}_K)$$

$$= \exp(-ct_K)\frac{\exp(-s_K)}{\left|\boldsymbol{\Sigma}_K\right|^{\frac{1}{2}}}\left|\tilde{\boldsymbol{\Sigma}}_K\right|^{\frac{1}{2}}\Pr\{\tilde{\mathbf{X}}(t_K)\in\boldsymbol{\Omega}_K\}$$

$$= \exp(-ct_K)\frac{\exp(-s_K)}{\left|\boldsymbol{\Sigma}_K\right|^{\frac{1}{2}}}\left|\tilde{\boldsymbol{\Sigma}}_K\right|^{\frac{1}{2}}\int_{\mathbf{x}(t_K)\in\boldsymbol{\Omega}_K} dF_{\tilde{\mathbf{X}}(t_K)}\left(\mathbf{x}(t_K)\right)$$

A4. Distribution of $\tilde{\mathbf{X}}(t_K)$

Based on the property of multivariate Gaussian random variables, $\tilde{\mathbf{X}}(t_K)$ is Gaussian with $\tilde{\mathbf{X}}(t_K) \sim N(\tilde{\boldsymbol{\mu}}(K), \tilde{\boldsymbol{\Sigma}}(K))$, where $\tilde{\boldsymbol{\mu}}(K) \equiv E(\tilde{\mathbf{X}}(t_K))$ and $\tilde{\boldsymbol{\Sigma}}(K) \equiv \mathrm{cov}(\tilde{\mathbf{X}}(t_K))$. Partition $\tilde{\boldsymbol{\Sigma}}_K, \tilde{\boldsymbol{\mu}}_K$ as following,

$$\tilde{\boldsymbol{\Sigma}}_K = \left[\begin{array}{c|c}\underset{(nK\times nK)}{\boldsymbol{\Sigma}11} & \underset{(nK\times n)}{\boldsymbol{\Sigma}12}\\ \hline \underset{(n\times nK)}{\boldsymbol{\Sigma}21} & \underset{(n\times n)}{\boldsymbol{\Sigma}22}\end{array}\right] \qquad \tilde{\boldsymbol{\mu}}_K = \left[\begin{array}{c}\underset{(nK\times 1)}{\boldsymbol{\mu}1}\\ \hline \underset{(n\times 1)}{\boldsymbol{\mu}2}\end{array}\right].$$

It is easy to see that $\tilde{\boldsymbol{\mu}}(K) = \boldsymbol{\mu}2$ and $\tilde{\boldsymbol{\Sigma}}(K) = \boldsymbol{\Sigma}22$.